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Quaternion-Based Rate/Attitude Tracking System with Application to Gimbal Attitude Control

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The paper deals with rate and attitude control of a rigid body. The discussed control system contains an inner velocity loop that tracks the desired rate command and an outer attitude loop that tracks the desired attitude command. The control algorithm uses quaternions as a measure of attitude errors to achieve tracking via instantaneous eigenaxis rotation. The paper discusses the stability of the new algorithm and presents its application to multiaxis gimbal attitude control.

I. Introduction

RAPID target acquisition, pointing, and tracking capabilities have become increasingly important to the success of many current and future spacecraft missions. The eigenaxis rotation via feed-forward command has been used for Apollo, Skylab, and the Shuttle and has been considered as a natural approach to a rapid, rotational maneuver.[1-3] Nonlinear optimal feedback control schemes,[4,5] a sliding mode control,[6] and a general nonlinear feedback control theory[7] have also been applied to the spacecraft attitude control problem. Recently, an eigenaxis rotation scheme via a quaternion-feedback regulator has been presented.[8] This paper extends the results of Wie et al.[8] and enables not only rest-to-rest maneuvers but also tracking via instantaneous eigenaxis rotation.

A closely related problem is the multiaxis gimbal attitude control. Gimbal systems used for radars and for homing devices also require rapid acquisition, pointing, and tracking. In these systems the control torques are not applied in the body-fixed axes, as in the rigid spacecraft case, but in the rotating axes of the gimbals. The paper shows how to use a rigid-body attitude control algorithm for the multiaxis gimbal case.

In Sec. II, we present the structure of a quaternion-based rate/attitude tracking system and discuss eigenaxis rotation in case of proportional (P) or proportional plus integral (PI) error quaternion controllers. In Sec. III, we discuss the stability of the proposed tracking system. With perfect cancellation of the gyroscopic torque and P or PI controller, we prove global stability and instantaneous eigenaxis rotation. Without cancellation of the gyroscopic torque, the stability proof is limited to the P controller, principal axis inertia, and axial symmetry. In Sec. IV, we consider the selection of the control system parameters. In Sec. V, we discuss the application of the proposed tracking system to gimbal attitude control. Three cases are considered: 1) tracking of three angles using a three-axis gimbal system, 2) tracking of two angles using a two-axis gimbal system, and 3) tracking of two angles using a three-axis gimbal system with constraint on the angle of the inner gimbal. Conclusions and discussions are given in Sec. VI.

II. Instantaneous Eigenaxis Rotation via Error Quaternion Feedback

In this section, the general case of a rigid body rotating under the influence of body-fixed torquing devices is considered. For simplicity, an ideal control torquer is assumed.

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Euler's Equations of Motion

Euler's equations[9] describe the rotational motion of a rigid body about body-fixed axes with origin at the center of mass. The following equations are associated with the general case in which the body-fixed control axes do not coincide with the principal axes of inertia:

$$J\dot{\omega}_B = \Omega_B J \omega_B + u \tag{1}$$

where $\omega_B = [\omega_{1B}\ \omega_{2B}\ \omega_{3B}]^T$ is the angular velocity vector of the rigid body, $u = [u_1\ u_2\ u_3]^T$ is the control torque vector, J is the inertia matrix, and $\Omega_B = [-\omega_B \times]$ is a skew-symmetric matrix defined by

$$\Omega_B = -\begin{bmatrix} 0 & -\omega_{3B} & \omega_{2B} \\ \omega_{3B} & 0 & -\omega_{1B} \\ -\omega_{2B} & \omega_{1B} & 0 \end{bmatrix} \tag{2}$$

Subscripts 1, 2, and 3 denote the body-fixed control axes.

It is assumed that the angular velocity components along the body-fixed control axes are measured by rate gyros and used to calculate the orientation.

Quaternion Kinematics

Euler's rotational theorem states that a rigid-body attitude can be changed from any given orientation to any other orientation by rotating the body about an axis called the Euler axis or eigenaxis. The quaternion defines the rigid-body attitude as an Euler-axis rotation.[10] The vector part of the quaternion (the first three components) indicates the direction of the Euler axis. The scalar part of the quaternion (the fourth component) is related to the rotation angle about the Euler axis. The four elements of the quaternion are defined as

$$q_i = c_i \sin(\phi/2); \qquad i = 1, 2, 3 \tag{3a}$$

$$q_4 = \cos(\phi/2) \tag{3b}$$

where ϕ is the magnitude of the Euler-axis rotation angle and (c_1, c_2, c_3) are the direction cosines of the Euler axis relative to a reference frame.

The quaternion kinematic differential equation[10] is described by

$$\dot{q}_e = \tfrac{1}{2}\Omega_e q_e + \tfrac{1}{2} q_{4e}\omega_e \tag{4a}$$

$$\dot{q}_{4e} = -\tfrac{1}{2}\omega_e^T q_e \tag{4b}$$

where $q_e = [q_{1e}\ q_{2e}\ q_{3e}]^T$ and q_{4e} define the error quaternion q^{BC} between the commanded (reference) quaternion and the body attitude quaternion, and Ω_e is defined by Eq. (2) with ω_B

replaced by $\omega_e = \omega_B - \omega_C$. Note that ω_e denotes the rigid-body angular velocity with respect to the commanded frame where ω_C denotes the commanded frame velocity. Using Eqs. (3), one can show that the quaternion satisfies the relation

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1 \tag{5}$$

Quaternion as a Measure of Attitude Errors

The quaternion $[q_{1B}, q_{2B}, q_{3B}, q_{4B}]$ defines the current body attitude with respect to an inertial frame. The quaternion $[q_{1C}, q_{2C}, q_{3C}, q_{4C}]$ defines the commanded attitude with respect to the same inertial frame. The attitude error between the current body attitude and commanded attitude is represented by q^{BC}. Since q^{BC} is calculated by

$$q^{BC} = q_C^* q_B \tag{6}$$

where q_C^* denotes the adjoint quaternion, then the error quaternion $q^{BC} \triangleq [q_{1e}, q_{2e}, q_{3e}, q_{4e}]$ is given by

$$\begin{bmatrix} q_{1e} \\ q_{2e} \\ q_{3e} \\ q_{4e} \end{bmatrix} = \begin{bmatrix} q_{4C} & q_{3C} & -q_{2C} & -q_{1C} \\ -q_{3C} & q_{4C} & q_{1C} & -q_{2C} \\ q_{2C} & -q_{1C} & q_{4C} & -q_{3C} \\ q_{1C} & q_{2C} & q_{3C} & q_{4C} \end{bmatrix} \begin{bmatrix} q_{1B} \\ q_{2B} \\ q_{3B} \\ q_{4B} \end{bmatrix} \tag{7}$$

For the special case of attitude regulation with respect to a reference frame specified at $t = 0$, the commanded quaternion is [0, 0, 0, 1]. In this case, the error quaternion coincides with the body attitude quaternion, i.e., $q_e = q_B$, $q_{4e} = q_{4B}$, and $\omega_e = \omega_B$.

Quaternion-Based Tracking System Structure

The proposed control system deals with rate and attitude control of a rigid body and consists of an inner angular rate loop and an outer attitude loop. The angular rate control loop is depicted in Fig. 1. The loop tracks the desired rate command by generating an angular acceleration in the direction of the angular velocity error. The magnitude of that angular acceleration is proportional to the magnitude of the velocity error.

Following Euler Eq. (1), we find that with the control torque

$$u = -\Omega_B J \omega_B + k_1 J \Delta\omega \tag{8}$$

the closed loop satisfies

$$\dot\omega_B = k_1 \Delta\omega \tag{9}$$

where $\Delta\omega \triangleq \omega_c - \omega_B$.

The attitude control loop is depicted in Fig. 2. The loop tracks the desired attitude by generating a rate command that is proportional to the attitude error and is directed along the instantaneous eigenaxis. The desired attitude is given in terms of a commanded quaternion. The attitude error is expressed

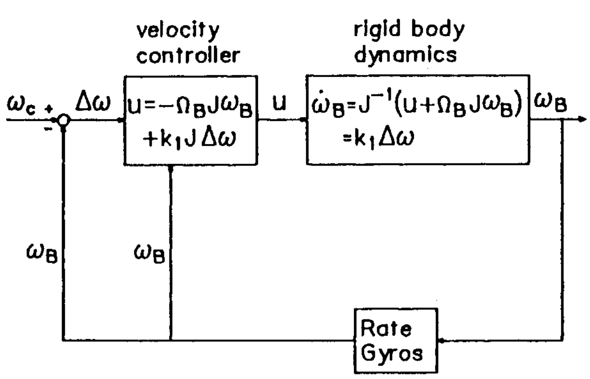

Fig. 1 Rate control loop.

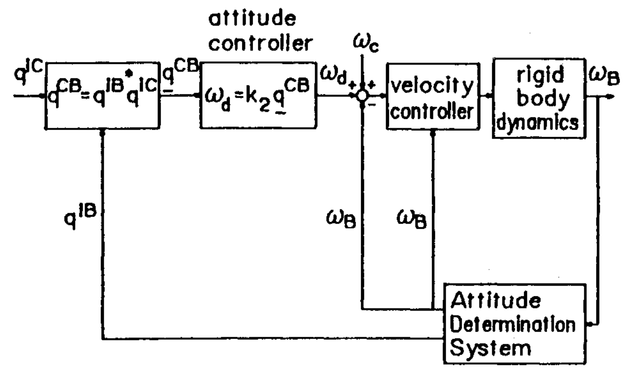

Fig. 2 Attitude control loop.

by the error quaternion between the commanded quaternion and the body attitude quaternion. The generated rate command is applied to the inner velocity loop.

Applying the control torque

$$u = -\Omega_B J \omega_B + k_1 J(\omega_c + \omega_d - \omega_B) \tag{10}$$

where

$$\omega_d = k_2 q^{CB} = -k_2 q^{BC} \tag{11}$$

and q^{CB}, q^{BC} denote the vector part of q^{CB} and q^{BC}, respectively, we realize that

$$u = -\Omega_B J \omega_B - dJ\omega_e - kJq_e \tag{12}$$

where $d \triangleq k_1$, $k \triangleq k_1 k_2$, $\omega_e \triangleq \omega_B - \omega_C$, and $q_e \triangleq q^{BC}$.

Equation (11) describes the case of a proportional error controller. The more general case of a proportional plus integral error controller is described by

$$\omega_d = -k_P q_e - k_I q_{eI} \tag{13}$$

where

$$\dot q_{eI} = q_e \tag{14}$$

The associated control torque is given by

$$u = -\Omega_B J \omega_B - dJ\omega_e - k_P J q_e - k_I J q_{eI} \tag{15}$$

Eigenaxis Rotation

The control torque selection presented in Eq. (15) guarantees a rotation about the instantaneous eigenaxis in the case where $\ddot\omega_C(t) \equiv 0$ and the initial conditions are $\omega_C(0) = \omega_B(0)$ and $\dot\omega_C(0) = \dot\omega_B(0)$. This statement is proved in the following theorem.

Theorem 1: Let u^* be defined by

$$u^* = -\Omega_B J \omega_B + \mu J \dot\omega_C - dJ\omega_e - k_P J q_e - k_I J q_{eI} \tag{16}$$

Let the initial conditions be $\omega_e(0) = 0$ and $\dot\omega_e = (0)$, then 1) the vectors $\omega_e \times q_e$ and $\dot\omega_e \times q_e$ are identically zero, and 2) $u = u^*$ implies eigenaxis rotation if and only if $\omega_e \times q_e = 0$ and $\dot\omega_e \times q_e = 0$. Observe that the conditions $\omega_e(0)$ and $\dot\omega_e(0) = 0$ are equivalent to $\omega_C(0) = \omega_B(0)$ and $\dot\omega_C(0) = \dot\omega_B(0)$, respectively.

Proof: 1) Let z_1 and z_2 be defined by $z_1 = \omega_e \times q_e$ and $z_2 = \dot\omega_e \times q_e$. Then, using Eqs. (1) and (4) with $u = u^*$, we find that

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}\Omega_e & I_3 \\ \tfrac{1}{2}[k_P(Q_e - q_{4e}I_3) + \dot\Omega_e] & -dI_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + (\mu - 1)\begin{bmatrix} 0 \\ q_e \times \ddot\omega_C \end{bmatrix} \tag{17}$$

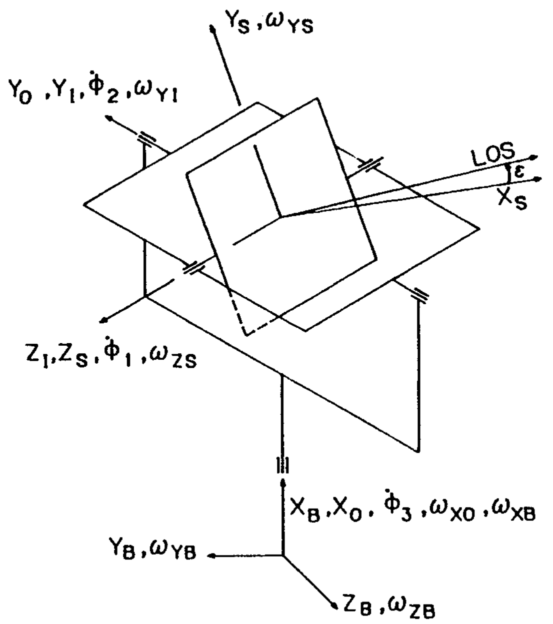

Fig. 3 Gimbal system definition.

where $Q_e \triangleq [-q_e \times]$, $\dot{\Omega}_e \triangleq [-\dot{\omega}_e \times]$, $\Omega_e \triangleq [-\omega_e \times]$, and I_3 is a 3×3 unit matrix. The derivation uses the fact that $\dot{\omega}_e \times \omega_e \equiv 0$. Assuming $\omega_e(0) = 0$, $\dot{\omega}_e(0) = 0$, and $\mu = 1$, then $z_1(0) = 0$, $z_2(0) = 0$, and hence, $z_1(t) \equiv 0$ and $z_2(t) \equiv 0$. Therefore, ω_e, $\dot{\omega}_e$, and q_e are collinear vectors.

2) *If:* If $\omega_e \times q_e = 0$, then $\omega_e(t) = \eta(t)q_e(t)$, where $\eta(t)$ is a scalar function.

Using this relation and Eq. (1) with $u = u^*$, we obtain

$$\dot{q}_e = ½ q_{4e}\eta q_e$$

and, thus,

$$q_e(t) = \left\{ \exp\left[\int_0^t ½ q_{4e}(\tau)\eta(\tau)\, \mathrm{d}\tau \right] \right\} q_e(0) = c_q(t)q_e(0) \quad (18)$$

Therefore,

$$\omega_e(t) = \eta(t)q_e(t) = \eta(t)c_q(t)q_e(0) = c_\omega(t)q_e(0) \quad (19)$$

If $\dot{\omega}_e \times q_e = 0$, then $\dot{\omega}_e(t) = \rho(t)q_e(t)$, where $\rho(t)$ is a scalar function. Hence,

$$\dot{\omega}_e(t) = \rho(t)c_q(t)q_e(0) = \dot{c}_\omega(t)q_e(0) \quad (20)$$

The relations $q_e = c_q(t)q_e(0)$, $\omega_e(t) = c_\omega(t)q_e(0)$, and $\dot{\omega}e(t) = \dot{c}_\omega(t)q_e(0)$ mean eigenaxis rotation.

Only if: If the rotation is about the eigenaxis, then

$$q_e(t) = c_q(t)q_e(0), \quad \omega_e(t) = c_\omega(t)q_e(0), \quad \dot{\omega}_e(t) = \dot{c}_\omega(t)q_e(0)$$

Therefore,

$$\omega_e \times q_e = [c_\omega q_e(0)] \times [c_q q_e(0)] \equiv 0$$

$$\dot{\omega}_e \times q_e = [\dot{c}_\omega q_e(0)] \times [c_q q_e(0)] \equiv 0$$

Remark 1: In case of a proportional error controller, i.e., $k_I = 0$, eigenaxis rotation can similarly be derived by dealing with z_1 only.

Remark 2: In the derivation of Eq. (17), the inclusion of the term $J\dot{\omega}_c$ in Eq. (16), i.e., $\mu = 1$, is equivalent to the requirement $\ddot{\omega}_c = 0$. If $k_I = 0$, then the inclusion of $J\dot{\omega}_c$ in Eq. (16) is equivalent to $\dot{\omega}_c = 0$. Therefore, in the case where $\omega_c(t) = (\omega_0 + \omega_1 t)1(t)$, eigenaxis rotation can be guaranteed only if $k_I \neq 0$.

Remark 3: In the case of eigenaxis rotation, the scalars $c_q(t)$ and $c_\omega(t)$ defined in Eqs. (18) and (19) satisfy the following equations:

$$\ddot{c}_\omega = -d\dot{c}_\omega - k_P\dot{c}_q - k_I c_q \quad (21a)$$

$$\dot{c}_q = ½ q_{4e}c_\omega \quad (21b)$$

$$\dot{q}_{4e} = -½ \| q_e(0) \|^2 c_\omega c_q \quad (21c)$$

If $k_I = 0$, then Eq. (21a) is replaced by

$$\dot{c}_\omega = -dc_\omega - k_P c_q \quad (21d)$$

III. Stability Analysis

In this section, we discuss the stability where the control torque is

$$u = -\mu_1\Omega_B J\omega_B - \mu_2\Omega_C J\omega_C + \mu_3 J\dot{\omega}_C - D\omega_e - K_P q_e - K_I q_{eI} \quad (22)$$

and the associated closed-loop equations are

$$J\dot{\omega}_B = (1-\mu_1)\Omega_B J\omega_B - \mu_2\Omega_C J\omega_C + \mu_3 J\dot{\omega}_C - D\omega_e - K_P q_e - K_I q_{eI} \quad (23a)$$

$$\dot{q}_{eI} = q_e \quad (23b)$$

$$\dot{q}_e = ½\Omega_e q_e + ½ q_{4e}\omega_e \quad (23c)$$

$$\dot{q}_{4e} = -½\omega_e^T q_e \quad (23d)$$

The stability is discussed in the following special cases.

Case 1: $\mu_1 = 1$, $\mu_2 = 0$, $\mu_3 = 1$, $K_I = 0$, i.e., proportional error controller with perfect cancellation of the gyroscopic torque.

Case 2: $\mu_1 = 0$, $\mu_2 = 1$, $\mu_3 = 1$, $K_I = 0$, i.e., proportional error controller without cancellation of the gyroscopic torque. The stability proof is limited to principal axis inertia and axial symmetry.

Case 3: $\mu_1 = 1$, $\mu_2 = 0$, $\mu_3 = 1$, i.e., proportional plus integral error controller with perfect cancellation of the gyroscopic torque. The stability proof is limited to rate commands characterized by $\omega_C(0) = \omega_B(0)$ and $\dot{\omega}_C(0) = \dot{\omega}_B(0)$.

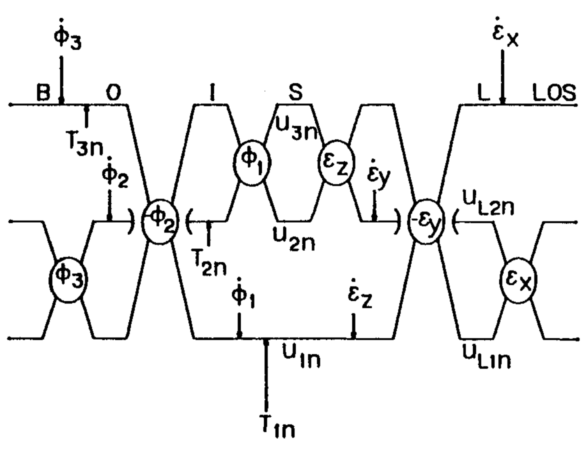

Fig. 4 Gimbal system piogram.

Case 1: Stability Analysis

Assuming that K_P^{-1} exists and $K_P^{-1}J$ is positive definite, we define the following Lyapunov function

$$V = \tfrac{1}{2}\omega_e^T K_P^{-1} J \omega_e + q_{1e}^2 + q_{2e}^2 + q_{3e}^2 + (q_{4e} - 1)^2$$

$$= \tfrac{1}{2}\omega_e^T K_P^{-1} J \omega_e + 2(1 - q_{4e}) \qquad (24)$$

Note that V is positive definite and asymptotically unbounded in ω_e.

The time derivative of V is given by

$$\dot V = \tfrac{1}{2}\dot\omega_e^T K_P^{-1} J\omega_e + \tfrac{1}{2}\omega_e^T K_P^{-1} J\dot\omega_e - 2\dot q_{4e}$$

Assuming that $K_P^{-1}J = (K_P^{-1}J)^T$, we can calculate $\dot V$ along the system trajectories as

$$\dot V = \omega_e^T K_P^{-1} J\dot\omega_e - 2\dot q_{4e} = -\omega_e^T K_P^{-1} D\omega_e \qquad (25)$$

Global stability is guaranteed if $K_P^{-1}D > 0$. A natural selection of D and K_P that guarantees this is

$$D = dJ \qquad K_P = k_P J \qquad (26)$$

where d and k_P are positive scalars. This selection of K_P also guarantees that K_P^{-1} exists and that $K_P^{-1}J$ is symmetric and positive definite.

Remark 1: The stability result is not limited to a principal axis inertia matrix, and the selection of the gains D and K_P is consistent with the definition of the control system parameters described in Sec. II.

Remark 2: If the term $J\dot\omega_c$ is not included in the control torque (i.e., $\mu_3 = 0$), the stability proof is limited to constant rate command.

Case 2: Stability Analysis

In this case, Eq. (23a) is reduced to

$$J\dot\omega_e = \Omega_e J\dot\omega_e - \bar D\omega_e - K_P q_e \qquad (27)$$

where

$$\bar D = D + \bar\Omega_C - \Omega_C J \qquad (28)$$

$$\bar\Omega_C = [-J\dot\omega_C \times], \qquad \Omega_C = [-\omega_C \times] \qquad (29)$$

Consider the Lyapunov function defined in Eq. (24) with $K_P^{-1}J = (K_P^{-1}J)^T$. The time derivative of V along the system trajectories is given by

$$\dot V = -\omega_e^T K_P^{-1}\bar D\omega_e + \omega_e^T K_P^{-1}\Omega_e J\omega_e \qquad (30)$$

Selecting the gain matrix K_P such that

$$K_P^{-1} = \alpha J + \beta I_3 \qquad (31)$$

where α and β are nonnegative scalars, we obtain

$$\omega_e^T K_P^{-1}\Omega_e J\omega_e = \omega_e^T(\alpha J + \beta I_3)\Omega_e J\omega_e = \alpha(J\omega_e)^T\Omega_e(J\omega_e)$$

$$+ \beta\omega_e^T\Omega_e J\omega_e \qquad (32)$$

Since $\Omega_e^T = -\Omega_e$, the first term in Eq. (32) is identically zero. Since Ω_e is a skew-symmetric matrix of the form described in Eq. (2), $\Omega_e\omega_e = \omega_e^T\Omega_e \equiv 0$, and the second term of Eq. (32) is identically zero. Equation (31) guarantees that K_P^{-1} exists and that $K_P^{-1}J$ is symmetric and positive definite.

Using Eq. (32), we find that

$$\dot V = -\omega_e^T K_P^{-1}\bar D\omega_e \qquad (33)$$

and global stability is guaranteed if $K_P^{-1}\bar D > 0$.

Using Eqs. (28) and (31), we obtain

$$K_P^{-1}\bar D = (\alpha J + \beta I_3)(D + \bar\Omega_C - \Omega_C J) = (\alpha J + \beta I_3)D + \alpha J\bar\Omega_C$$

$$+ \beta\bar\Omega_C - \alpha J\Omega_C J - \beta\Omega_C J$$

and, therefore,

$$\omega_e^T K_P^{-1}\bar D\omega_e = \omega_e^T[(\alpha J + \beta I_3)D + \alpha J\bar\Omega_C - \beta\Omega_C J]\omega_e \qquad (34)$$

In the case where the body-fixed axes coincide with the principal axes, the inertia matrix is given as

$$J = \mathrm{diag}[J_1 \; J_2 \; J_3] \qquad (35)$$

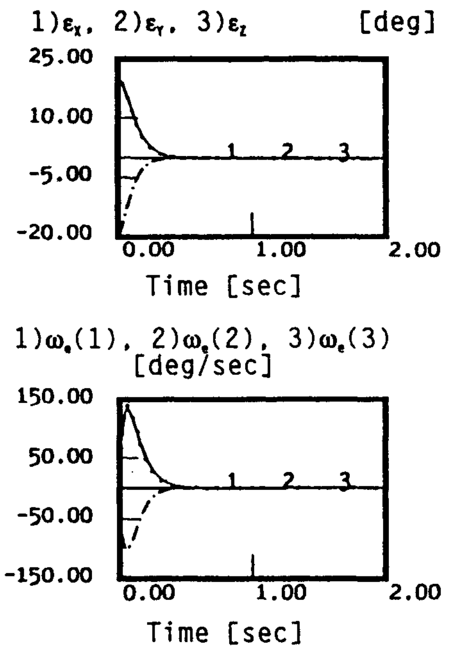

Fig. 5a Attitude and rate errors in case 1.

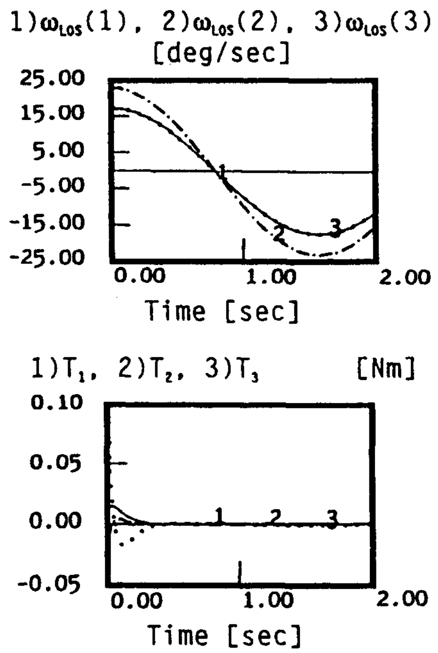

Fig. 5b Line-of-sight rates and control torques in case 1.

For this case, the matrix $[\alpha J\bar{\Omega}_C - \beta\bar{\Omega}_C J]$ has the form

$$[\alpha J\bar{\Omega}_C - \beta\Omega_C J] = \begin{bmatrix} 0 & (\alpha J_1 J_3 - \beta J_2)\omega_{3C} & (-\alpha J_1 J_2 + \beta J_3)\omega_{2C} \\ (-\alpha J_2 J_3 + \beta J_1)\omega_{3C} & 0 & (\alpha J_1 J_2 - \beta J_3)\omega_{1C} \\ (\alpha J_2 J_3 - \beta J_1)\omega_{2C} & (-\alpha J_1 J_3 + \beta J_2)\omega_{1C} & 0 \end{bmatrix}$$

This matrix is a skew-symmetric matrix if the following relation holds

$$\frac{J_3 - J_2}{J_2^2 J_3^2} + \frac{J_1 - J_3}{J_1^2 J_3^2} + \frac{J_2 - J_1}{J_1^2 J_2^2} = 0 \tag{36}$$

In case of axial symmetry (i.e., $J_1 = J_2$ or $J_1 = J_3$ or $J_2 = J_3$), Eq. (36) is always satisfied and Eq. (33) is reduced to

$$\dot{V} = -\omega_e^T[(\alpha J + \beta I_3)D]\omega_e \tag{37}$$

Global stability is guaranteed if $(\alpha J + \beta I_3)D > 0$. The matrix $(\alpha J + \beta I_3)D$ is positive definite if D is selected such that $D = dJ$.

Remark 3: The Lyapunov function defined in Eq. (24) considers the negative quaternion-feedback gain and the equilibrium point $\omega_e = 0$, $q_e = 0$, $q_{4e} = 1$. The Lyapunov function associated with the positive quaternion-feedback gain and the equilibrium point $\omega_e = 0$, $q_e = 0$, $q_{4e} = -1$ is described by

$$V = ½\omega_e^T K_P^{-1} J\omega_e + q_{1e}^2 + q_{2e}^2 + q_{3e}^2 + (q_{4e} + 1)^2$$

$$= ½\omega_e^T K_P^{-1} J\omega_e + 2(1 + q_{4e}) \tag{38}$$

In this case, the term $-K_P q_e - K_I q_{eI}$ that appeared in Eq. (22) is replaced by $K_P q_e + K_I q_{eI}$.

To guarantee the shortest angular path, the sign of the quaternion-feedback gain is determined by the initial value of q_{4e}. The corresponding term in the control torque equation is $-(K_P q_e + K_I q_{eI})\text{sign}[q_{4e}(0)]$.

Case 3: Stability Analysis

In this case, the stability analysis is reduced to that of Eqs. (21). Consider the following Lyapunov function,

$$V = ½x^T P x \tag{39}$$

where

$$x = [x_1 \; x_2 \; x_3 \; (x_4 - 1)]^T$$

$$x_1 = c_\omega \quad x_2 = \dot{c}_\omega \quad x_3 = c_q \quad x_4 = q_{4e}$$

$$P = \begin{bmatrix} 1 & \frac{1}{d} & \frac{k_P}{d} & 0 \\ \frac{1}{d} & \frac{(1+\alpha)}{d^2} & \frac{k_P(1+\alpha)}{d^2} & 0 \\ \frac{k_P}{d} & \frac{k_P(1+\alpha)}{d^2} & \frac{2k_I}{d} + \frac{k_P^2(1+\alpha)}{d^2} & 0 \\ 0 & 0 & 0 & \frac{2k_I}{d\|q_e(0)\|^2} \end{bmatrix} \tag{40}$$

with $\alpha > 0$.

The time derivative of V along the system trajectories is given by

$$\dot{V} = \frac{-\alpha}{d}x_2^2 - \frac{k_P k_I(1+\alpha)}{d^2}x_3^2 \tag{41}$$

Observe that $\dot{V} < 0$ since $x_3(0) = 1$ and, therefore, global stability is guaranteed if $P > 0$. The matrix P is positive definite if $d > 0$, $k_P > 1$, $k_I > 0$, and $\alpha > 0$.

Remark 4: In the derivation of Eqs. (21), the inclusion of the term $J\dot{\omega}_c$ in the control torque is equivalent to the requirement that $\ddot{\omega}_c = 0$. Therefore, if we select $\mu_3 = 0$ in Eq. (22), then the stability result is limited to a constant angular acceleration command.

IV. Selection of the Tracking System Parameters

The selection of the gain is based on the assumption that eigenaxis rotation exists. Let the quaternion-feedback gains and the damping gain satisfy $K_P = k_P J$, $K_I = k_I J$, and $D = dJ$, where k_P, k_I, and d are positive scalars. Let λ be a unit vector along the eigenaxis. Then assuming eigenaxis rotation, the error quaternion q_e and the angular rate ω_e satisfy the relations $q_e = [\sin(\phi_e/2)]\lambda$ and $\omega_e = \dot{\phi}_e\lambda$, where ϕ_e is the eigenangle. Also assuming that ω_e is small enough to allow the gyroscopic term to be neglected, or that the gyroscopic torque can be counteracted if its effect is significant, Eq. (23a) can be approximated by

$$\{\ddot{\phi}_e + d\dot{\phi}_e + ½[k_P\cos(\phi_e/2)]\dot{\phi}_e + k_I\sin(\phi_e/2)\}J\lambda = 0 \tag{42}$$

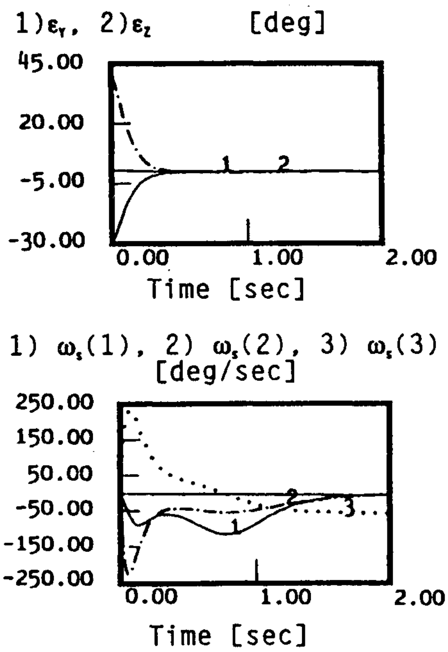

Fig. 6a Attitude errors and inner gimbal rates in case 2.

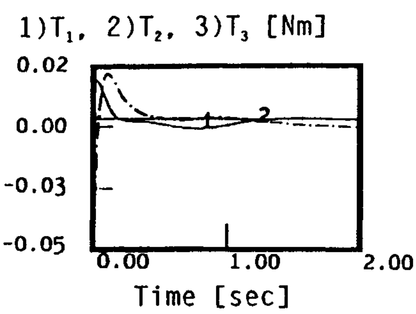

Fig. 6b Control torques in case 2.

Since $J\lambda \neq 0$, Eq. (42) is reduced to

$$\dddot{\phi}_e + d\ddot{\phi}_e + ½[k_P \cos(\phi_e/2)]\dot{\phi}_e + k_I \sin(\phi_e/2) = 0 \quad (43)$$

For the purpose of selecting gains, we may approximate $\sin(\phi_e/2)$ by $\phi_e/2$ and $\cos(\phi_e/2)$ by 1, where $\phi_e \leq 90$ deg. We then have a linear third-order differential equation

$$\dddot{\phi}_e + d\ddot{\phi}_e + ½k_P\dot{\phi}_e + ½k_I\phi_e = 0 \quad (44)$$

that can be characterized by

$$(s + a)(s^2 + 2\zeta\omega_n s + \omega_n^2) = 0$$

where a, ζ, and ω_n are positive scalars.

Then k_P, k_I, and d satisfy the following equations

$$k_P = 2\omega_n^2 + 4a\zeta\omega_n, \quad k_I = 2a\omega_n^2, \quad d = 2\zeta\omega_n + a \quad (45)$$

Observe that for the proportional error controller case, $k_I = 0$ and, hence, $a = 0$. Also observe that in this case Eq. (44) is reduced to

$$\ddot{\phi}_e + d\dot{\phi}_e + ½k_P\phi_e = 0 \quad (46)$$

Remark 1: Using Eqs. (45), the matrix P defined by Eq. (40) is positive definite if $\omega_n \geq 0.5$, $\zeta > 0$, $a > 0$, and $\alpha > 0$. Thus, the selection of k_P, k_I, and d according to Eqs. (45), where $\omega_n \geq 0.5$, $\zeta > 0$, and $a > 0$, guarantees stability.

Remark 2: The effect of the PI error controller can be demonstrated where the control torque defined by Eq. (22) is characterized with $\mu_1 = 1$ and $\mu_2 = \mu_3 = 0$. Assuming that $D = dJ$, $K_P = k_P J$, and $K_I = k_I J$, then the Euler equation is reduced to

$$\dot{\omega}_e + d\omega_e + k_P q_e + k_I q_{eI} = -\dot{\omega}_C \quad (47)$$

Since $\ddot{\omega}_C = 0$ can guarantee eigenaxis rotation, we consider rate commands of the form $\omega_C(t) = (\omega_O + \omega_1 t)1(t)$. In this case, Eq. (47) may be approximated by

$$\{s^2 + ds + ½[k_P + (k_I/s)]\}\phi_e(s) = -s\omega_C(s)$$

and, therefore, the transfer function $\phi_e(s)/\omega_C(s)$ is given by

$$\frac{\phi_e(s)}{\omega_C(s)} = -\frac{s^2}{s^3 + ds^2 + ½k_P s + ½k_I}$$

Thus, zero steady-state eigenangle is guaranteed for the considered rate commands. If the P error controller is used, then zero steady-state eigenangle can be achieved only for constant rate command.

V. Application to Gimbal Attitude Control

The gimbal system is depicted in Fig. 3. The system contains yaw, pitch, and roll gimbals. The inner (yaw) gimbal is the stabilized platform on which the sensor and three rate gyros are mounted. The sensor measures the angular deviation ϵ of the boresight x_s from the target line of sight (LOS).

The gimbal system just described has the following equations of moments:

$$T_s = J_s\dot{\omega}_s + \omega_s \times J_s\omega_s + \omega_s \times H_s \quad (48a)$$

$$T_I = J_I\dot{\omega}_I + \omega_I \times J_I\omega_I + [-\phi_1]T_s \quad (48b)$$

$$T_O = J_O\dot{\omega}_O + \omega_O \times J_O\omega_O + [-\phi_2]T_I \quad (48c)$$

where

$$\omega_O = [\phi_3]\omega_B + [\dot{\phi}_3 \ 0 \ 0]^T \quad \omega_I = [\phi_2]\omega_O + [0 \ \dot{\phi}_2 \ 0]^T$$

$$\omega_s = [\phi_1]\omega_I + [0 \ 0 \ \dot{\phi}_1]^T \quad (49)$$

and where ω_s, ω_I, and ω_O denote the angular velocities of each gimbal system and are obtained using the piogram[11] of Fig. 4; J_s denotes the inner gimbal inertia matrix defined with respect to the gimbal axes; J_I denotes the intermediate gimbal inertia matrix defined with respect to the gimbal axes where the inertia of the inner gimbal is not included; J_O denotes the outer gimbal inertia matrix defined with respect to the gimbal axes where the inertia of the inner gimbals are not included; H_{XS}, H_{YS}, and H_{ZS} denote the constant angular momentum of the yaw, pitch, and roll rate gyros, respectively; T_S, T_I, and T_O denote the control torques acting on each gimbal; $[\phi_3]$, $[\phi_2]$, and $[\phi_1]$

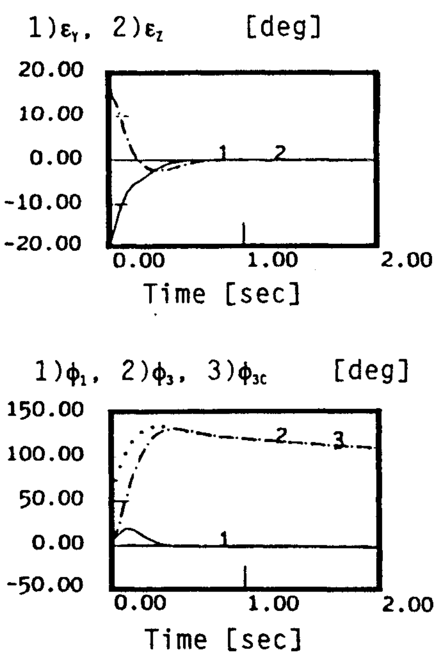

Fig. 7a Attitude errors and gimbal angles in case 3.

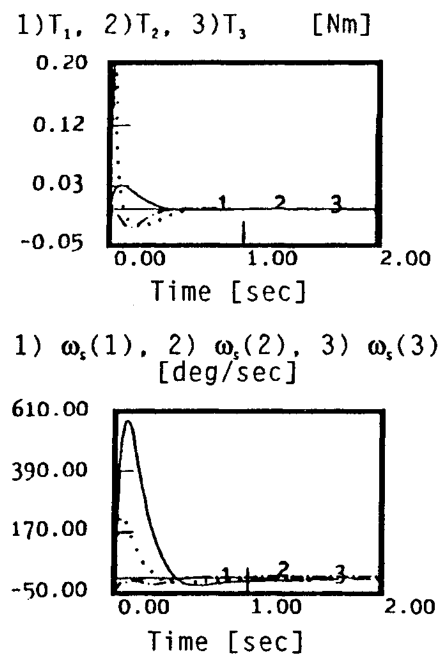

Fig. 7b Control torques and inner gimbal rates in case 3.

denote the rotation matrices from B to O, O to I, and I to S, respectively; and $[-\phi_i] \triangleq [\phi_i]^T$.

Assuming that the gimbal rotation axes intersect in one point and that $\{XS, YS, ZS\}$, $\{XI, YI, ZI\}$, and $\{XO, YO, ZO\}$ are the principal axes, the control torques needed to rotate the gimbals are

$$T_{XO} = J_{XX}\dot{\omega}_{XO} + J_{XY}\omega_{YI} + T_{DXO} \quad (50a)$$

$$T_{YI} = J_{XY}\dot{\omega}_{XO} + J_{YY}\dot{\omega}_{YI} + T_{DYI} \quad (50b)$$

$$T_{ZS} = J_{ZZ}\dot{\omega}_{ZS} + T_{DZS} \quad (50c)$$

where

$$J_{XX} = J_{XO} + J_{XI}C^2\phi_2 + J_{ZI}S^2\phi_2 + (J_{XS}C^2\phi_1 + J_{YS}S^2\phi_1)C^2\phi_2 \quad (51a)$$

$$J_{YY} = J_{YI} + J_{XS}S^2\phi_1 + J_{YS}C^2\phi_1 \quad (51b)$$

$$J_{ZZ} = J_{ZS} \quad (51c)$$

$$J_{XY} = (J_{XS} - J_{YS})S\phi_1 C\phi_1 C\phi_2 \quad (51d)$$

where T_{DXO}, T_{DYI}, and T_{DZS} are coupling terms, and $C\phi_i$ and $S\phi_i$ denote $\cos\phi_i$ and $\sin\phi_i$, respectively.

The application of the suggested tracking system is considered in three cases: 1) tracking of three angles using a three-axis gimbal system, 2) tracking of two angles using a two-axis gimbal system, and 3) tracking of two angles using a three-axis gimbal system with constraint on the angle of the inner gimbal. In all cases, the control torque associated with the inner gimbal system has the form

$$u = -\omega_s \times J_s\omega_s - \omega_s \times H_s - dJ_s\omega_e - kJ_s\,\text{sign}[q_{4e}(0)]q_e \quad (52)$$

where

Case 1:

$$q_e = q^{SLOS}(\epsilon_X, \epsilon_Y, \epsilon_Z)$$

$$\hat{\omega}_{LOS} = \hat{\omega}_{LOS}(\omega_S, \epsilon_X, \epsilon_Y, \epsilon_Z, \hat{\dot{\epsilon}}_X, \hat{\dot{\epsilon}}_Y, \hat{\dot{\epsilon}}_Z)$$

$$\omega_e = \omega_s - [\epsilon_Z][\epsilon_Y][\epsilon_X]\hat{\omega}_{LOS}$$

where $\hat{(\cdot)}$ denotes estimated value of $(\cdot)$.

Case 2:

$$q_e = q^{SL}(\epsilon_Y, \epsilon_Z)$$

$$\hat{\omega}_L = \hat{\omega}_L(\omega_S, \epsilon_Y, \epsilon_Z, \hat{\dot{\epsilon}}_Y, \hat{\dot{\epsilon}}_Z)$$

$$\omega_e = \omega_s - [\epsilon_Z][\epsilon_Y]\hat{\omega}_L$$

Case 3: As in case 2 for the control of the yaw and pitch gimbals but with an additional control for the roll gimbal:

$$T_3 = -d_1 J_{XX}(\hat{\dot{\phi}}_3 - \hat{\dot{\phi}}_{3C}) - k_1 J_{XX}\,\text{sign}\left\{\cos\left[(\phi_3 - \phi_{3C})/2\right]\right\} \cdot \sin\left[(\phi_3 - \phi_{3C})/2\right] \quad (53)$$

where

$$\phi_{3C} = \tan^{-1}\left[\frac{S\phi_1 C\phi_3 + C\phi_1 S\phi_2 S\phi_3}{C\phi_1 S\phi_2 C\phi_3 - S\phi_1 S\phi_3}\right] \quad (54)$$

is the command to the roll gimbal defined by the requirements

$$T^{BS}\begin{bmatrix}1\\0\\0\end{bmatrix}_s = T^{B\bar{S}}\begin{bmatrix}1\\0\\0\end{bmatrix}_{\bar{s}}, \qquad \bar{\phi}_1 = 0$$

where T^{BS} denotes the transformation matrix from S to B and is defined by ϕ_1, ϕ_2, ϕ_3; and $T^{B\bar{S}}$ denotes the transformation matrix from $\bar{S}$ to B and is defined by $\bar{\phi}_2$, $\bar{\phi}_3$, where $1_{X\bar{S}} \equiv 1_{XS}$ and $1_{X\bar{S}}$, 1_{XS} are unit vectors along the axes $X\bar{S}$, XS, respectively.

The requirements correspond to zero rotation constraint on the inner gimbal using the outer gimbal rotation $\bar{\phi}_3 = \phi_{3C}$ to guarantee target LOS tracking. This constraint is due to mechanical limitations of the inner gimbal that carries the sensors.

Let T_1, T_2, and T_3 denote the control torques in the yaw, pitch, and roll rotation axes, respectively. Using the gimbal system piogram, we can transfer the angular accelerations

$$T_{1n} \triangleq T_1/J_{ZZ}, \qquad T_{2n} \triangleq T_2/J_{YY}, \qquad T_{3n} \triangleq T_3/J_{XX} \quad (55)$$

to the S system in case 1, and to the L system in cases 2 and 3. In case 1, the transferred angular acceleration should generate the required angular acceleration u_{1n}, u_{2n}, u_{3n}, where $u_n = J_s^{-1}u$ and u is defined by Eq. (52). In cases 2 and 3, the transferred angular acceleration should generate the required angular acceleration u_{L1n}, u_{L2n}, where

$$u_{L1n} = u_{1n}C\epsilon_Y + (u_{3n}C\epsilon_Z + u_{2n}S\epsilon_Z)S\epsilon_Y \quad (56a)$$

$$u_{L2n} = -u_{3n}S\epsilon_Z + u_{2n}C\epsilon_Z \quad (56b)$$

In these cases, we use only two components of the control torque to track the LOS since the roll angle of the LOS is not relevant.

Therefore, the required control torques T_1, T_2, and T_3 are calculated using Eqs. (55) and one of the following case-dependent equations.

Case 1:

$$T_{1n} = u_{1n} - (u_{3n}C\phi_1 - u_{2n}S\phi_1)\tan\phi_2 \quad (57a)$$

$$T_{2n} = u_{3n}S\phi_1 + u_{2n}C\phi_1 \quad (57b)$$

$$T_{3n} = (u_{3n}C\phi_1 - u_{2n}S\phi_1)/C\phi_2 \quad (57c)$$

Case 2:

$$T_{1n} = [u_{L1n} - u_{L2n}\tan(\phi_1 + \epsilon_Z)S\epsilon_Y]/C\epsilon_Y \quad (58a)$$

$$T_{2n} = u_{L2n}/C(\phi_1 + \epsilon_Z) \quad (58b)$$

Case 3:

$$T_{1n} = [\bar{u}_{L1n} - \bar{u}_{L2n}\tan(\phi_1 + \epsilon_Z)S\epsilon_Y]C\epsilon_Y \quad (59a)$$

$$T_{2n} = \bar{u}_{L2n}/C(\phi_1 + \epsilon_Z) \quad (59b)$$

where

$$\bar{u}_{L1n} = u_{L1n} + T_{3n}[C\phi_2 C(\phi_1 + \epsilon_Z)S\epsilon_Y + S\phi_2 C\epsilon_Y] \quad (60a)$$

$$\bar{u}_{L2n} = u_{L2n} - T_{3n}C\phi_2 S(\phi_1 + \epsilon_Z) \quad (60b)$$

and T_{3n} is determined by Eq. (53).

Table 1 Initial conditions and LOS angular velocity

Case	$\phi(0)$, deg	$\epsilon(0)$, deg	$\omega_{LOS}(t)$, rad/s
1	[0, 0.0]	[20, 20, 20]	[0.3 cos 2t, 0.4 cos 2t, 0.3 cos 2t]
2	[5, 5]	[30, 30, 40]	[0.7, 0.6, 0.7]
3	[5, 5, 5]	[15, 20, 15]	[0.1, 0.1, 0.1]

Using the control algorithms of this section, the results presented in Figs. 5-7 correspond to the following numerical data

$$J_S = \text{diag}[9.2 \times 10^{-4},\ 9.2 \times 10^{-4},\ 9.2 \times 10^{-4}]\text{kg-m}^2$$

$$J_I = \text{diag}[1.3 \times 10^{-3},\ 1.9 \times 10^{-3},\ 1.1 \times 10^{-3}]\text{kg-m}^2$$

$$J_O = \text{diag}[6.6 \times 10^{-3},\ 6.6 \times 10^{-3},\ 6.2 \times 10^{-3}]\text{kg-m}^2$$

$$H_S = [3.48 \times 10^{-3},\ 3.48 \times 10^{-3},\ 3.48 \times 10^{-3}]^T\text{kg-m}^2\ \text{rad/s}$$

$$d = 33, \quad k = 533, \quad d_1 = 16, \quad k_1 = 130$$

The initial conditions and the LOS angular velocity are defined in Table 1. Observe that in cases 2 and 3 we do not control ϵ_X and, therefore, the LOS velocity is not necessarily tracked. In all cases, we have a settling time of 0.5 s and excellent performance.

VI. Conclusions

The quaternion-based rate/attitude tracking system of this paper enables both rate and attitude tracking while using the proportional or proportional plus integral error quaternion controller.

Since eigenaxis rotation provides the shortest angular path, the proposed controller may provide a simple solution for large-angle reorientation and the tracking of future spacecraft. The application of the controller to gimbal attitude control also extends the results to a more general case where the control torques are applied to nonfixed axes.

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